J. Appl. Maths Mechs, Vol. 55, No. 2, pp. 248-254, 1991
Printed in Great Britain

0021-8928/91 $15.00.0+00

THE PROBLEM OF THE JUSTIFICATION OF VARIATIONAL FORMULATIONS OF THE BOUNDARY ELEMENT METHOD*

V.YA. TERESHCHENKO

An investigation is made of the convergence of boundary element approximations of variational solutions of the boundary-value problems of linear elasticity theory, using double- and simple-layer potentials (DLP, SLP) /1, 2/. Auxiliary propositions are proved, concerning the basis property of the approximating sequence of potentials. The results may also be used to justify variational formulations of the boundary element method (BEM) is applied to the solution of other second-order elliptic boundary-value problems.

In /1, 2/ we proposed formulations of the BEM in which solutions of variational problems for the boundary functionals (BFs) and generalized Trefftz functionals (GTFs) of the boundary-value problems of linear elasticity theory are approximated using DLPs and SLPs; the densities of the latter are interpolated from the nodal values of the displacements (direct formulation), or from the nodal stresses (dual formulation). The nodal values of the coefficients of the Ritz approximations to the solutions of the variational problems are also determined from a system of discrete boundary equations. The interpolation polynomial, whose coefficients are determined in terms of the nodal displacements, is complete /3/; in physical terms, completeness means that the field of constant strains can be approximated, i.e., as the partition into boundary elements (BEs) becomes progressively finer, the strained state inside each element will approach a state of constant strain; all this in combination guarantees convergence of the approximate solution.

Thus, the Ritz approximations for each partition into BEs are essentially the same as a sequence of DLPs (or SLPs), in which the (unknown) vector density is a complete polynomial. To prove the convergence of the Ritz BE-approximations to the solution of the boundary-value problem, it is then sufficient to prove that the sequence of DLPs (or SLPs) is complete in the energy norm of the problem or in the equivalent Sobolev norm /1/ - subject to conditions that guarantee convergence of the BE-approximations as the elements decrease in size, namely, completeness of the BE basis functions and the compatibility of the elements /3/. In this connection, it should be emphasized that the variational formulations of the BEM proposed in /1, 2/ are $(m-1)$-dimensional versions of the finite element method (FEM) in the Ritz formulation, and so it is legitimate to apply well-known criteria for completeness of the finite element basis functions and the compatibility of the elements, which imply convergence of the FEM in its Ritz variational formulation (see, e.g., /4-6/): completeness of the system of basis functions in the FEM is achieved by taking a complete interpolation polynomial; the compatibility of the elements means that the interpolating function and its derivatives of order up to and including $q-1$ must be continuous across the common boundaries of adjacent elements, where q is the highest order of the derivatives involved in the functional of the variational formulation.

Some propositions aimed at justifying the variational formulations of the BEM were stated in /1, 2/; these propositions will be proved here.

1. To simplify the discussion (retaining our previous notation in /1, 2/), we shall consider DLPs and SLPs with scalar density:

$$\beta_n(x) = c_1 \int\limits_{\Delta s_n(y)} \partial_{\nu_\Delta}\Gamma(x, y)\,\varphi_n(y)\,ds_n(y) \tag{1.1}$$
$$\gamma_n(x) = c_2 \int\limits_{\Delta s_n(y)} H(x, y)\,t_n(y)\,ds_n(y), \quad x \in G_\Delta$$

(henceforth the constants c_1, c_2 will be omitted). Here

$$S_\Delta = \bigcup_n \Delta s_n \tag{1.2}$$

*Prikl.Matem.Mekhan., 55, 2, 309-316, 1991 is the discrete boundary, bounding a domain $G_\Delta \subset G$, where G is an m-dimensional bounded domain with sufficiently smooth boundary S; Δs_n are the boundary elements. It is assumed that $G_\Delta \to G$ as $\operatorname{diam} \Delta s_n \to 0$ (or $S_\Delta \equiv S$); $\Gamma(x, y)$, $x, y \in \bar{G}_\Delta = G_\Delta + S_\Delta$, is Green's function of a second-order elliptic differential operator with constant coefficients:

$$A\Gamma(x, y) = 0,\ x, y \in G_\Delta\ (x \neq y);\ \Gamma(x, y)|_{y \in S_\Delta} = 0 \tag{1.3}$$

and $\partial_{\nu_\Delta} = \partial/\partial\nu_\Delta$ is differentiation in the direction of the outward conormal (normal)) to the surface S_Δ; $H(x, y)$, $x, y \in \bar{G}_\Delta$ is the "Neumann function" of the operator:

$$AH(x, y) = 0,\ x, y \in G_\Delta\ (x \neq y);\ \partial_{\nu_\Delta} H(x, y)|_{y \in S_\Delta} = 0 \tag{1.4}$$

- the existence of these functions is postulated, Formulae (1.3) and (1.4) have been rigorously proved for a piecewise smooth boundary S_Δ /4/;

$$\varphi_n(y(\eta)) = \sum_k \Phi_{nk}\psi_k(\eta), \quad \eta \in \Delta s_n, \quad n = 1, \ldots, N \tag{1.5}$$

are interpolating functions defined in terms of the nodal values Φ_{nk} (k is the number of a node) of the potentials β_n and basis functions ψ_k corresponding to an interpolating polynomial of order p /3/, where $y(\eta)$ defines the relationship between the local coordinate (η) and the global coordinate (y) of the points of the BE ΔS_n;

$$t_n(y(\eta)) = \sum_k T_{nk}\psi_k'(\eta), \quad \eta \in \Delta s_n, \quad n = 1, \ldots, N \tag{1.6}$$

are interpolating functions defined in terms of the nodal values T_{nk} of the normal derivative of the potentials γ_n and basis functions ψ_k'; in particular, one can take $\psi_k' = \partial_{\nu_n}\psi_k$, where $\nu_n(\eta)$ is the outward conormal (normal) at points $\eta \in \Delta s_n$. Throughout this paper, unless otherwise stated, summation over k runs from $k = 1$ to $k = K$, and summation over n, from $n = 1$ to $n = N$.

The potentials (1.1) will be used for BE-approximation of solutions in the direct and dual formulations of the variational problems for the BFs and GTFs.

A sufficient condition for the existence of the integrals in (1.1) is that the densities be continuous at the points of Δs_n (assuming that the BEs themselves are sufficiently smooth) /7/; this imposes certain conditions on the smoothness of the interpolating polynomials (in the sense of their order).

A sufficient condition for the interpolating functions

$$\sum_n \varphi_n(y), \quad \sum_n t_n(y), \quad y \in S_\Delta$$

given by (1.5) and (1.6), to be continuous on the boundary S_Δ, $\forall N$ is that the BEs Δs_n be compatibile (see above); this in turn is ensured while the system of equations of the BEM is being set up, by taking the nodal values Φ_{nk} (in the direct formulation) and T_{nk} (in the dual formulation) to be equal at common nodes of adjacent elements /1, 2/. Thus, by construction, the approximations

$$\sum_n \beta_n|_{S_\Delta} = \sum_n \varphi_n(y) \tag{1.7}$$

$$\sum_n \partial_{\nu_n}\gamma_n|_{S_\Delta} = \sum_n t_n(y) \tag{1.8}$$

are continuous and their first derivatives have a finite number of points and curves of discontinuity, so that they belong to the space $W_2^1(S_\Delta)$, which in this case is a subset of the space of continuous functions $C(S_\Delta)$ - the set of all functions whose first derivatives with respect to $y \in S_\Delta$ are elements of the space $L_2(S_\Delta)$ of square-integrable functions over S_Δ (see (1.2)) (this definition is topologically equivalent to the usual definition of the Sobolev class $W_2^1(S)$ see /8/).

In problems of linear elasticity theory (the isotropic case) the function $\Gamma(x, y)$ corresponds to Green's tensor of the first problem of statics, the function $H(x, y)$ corresponds to Green's tensor of the second problem of statics; the density of the potential defined by the first (second) formula in (1.1) is interpolated from the nodal displacements (stresses); the corresponding vector potentials then describe the field of displacements of the points $x \in G_\Delta$ of the elastic medium /1, 2/.

We will now establish the basis property of the sequences of potentials

$$\{\beta_n(x)\}_{n=1,\ldots,N},\ \{\gamma_n(x)\}_{n=1,\ldots,N},\ x\in S_\Delta \tag{1.9}$$

For $x\in\Delta s_n$ the potentials (1.1) will be treated as linear integral operators with domain of definition $C(\Delta s_n)$:

$$\beta_n(x)\equiv R(\varphi_n)(x),\ \gamma_n(x)\equiv R'(t_n)(x),\ x\in\Delta s_n$$

and range in $L_2(\Delta s_n)$.

Let us recall some properties of boundary potentials that will be needed in what follows /9, 10/.

An SLP has a singularity $O(r^{-1})$, $r=|x-y|$, $x,y\in S$, and accordingly a DLP has a singularity $O(r^{-2})$; consequently, an SLP is an integral with a weak singularity if S is a two-dimensional boundary and a singular integral if S is a one-dimensional boundary. Whatever the dimensionality of S, a DLP is a singular integral. The integral operators R' and R generated by a SLP and a DLP have corresponding properties.

It is known /10/ that an integral operator with weak singularity is completely continuous and therefore bounded in L_2. A singular integral operator is bounded in L_2, subject to certain conditions on the characteristic of the operator as a singular integral /10/. Thus, the integral operators R and R', applied to the respective densities, are bounded in $L_2(\Delta s_n)$:

$$\begin{gathered}\|R(\varphi_n)\|_{L_2(\Delta s_n)}\leqslant c\|\varphi_n\|_{L_2(\Delta s_n)}\\ \|R'(t_n)\|_{L_2(\Delta s_n)}\leqslant c'\|t_n\|_{L_2(\Delta s_n)}\\ c=\|R\|_{L_2(\Delta s_n)}<\infty,\qquad c'=\|R'\|_{L_2(\Delta s_n)}<\infty\end{gathered} \tag{1.10}$$

If $\varphi_n=0$ ($\Phi_{nk}=0$) and $t_n=0$ ($T_{nk}=0$), we have $\beta_n=0$ and $\gamma_n=0$, $n=1,2,\ldots$, respectively; consequently,

$$R(0)=0,\ R'(0)=0 \tag{1.11}$$

The sequence (1.9) will be complete in $L_2(S_\Delta)$ if the ranges of the operators

$$\sum_n R_n\,(R_n\equiv R(\varphi_n)),\quad \sum_n R_n'\,(R_n'\equiv R'(t_n))$$

are dense in $L_2(S_\Delta)$.

To prove this, we need certain regularity properties of boundary potentials in terms of the space of Hölder continuous functions $C^{0,\alpha}$, $0<\alpha\leqslant 1$, in a bounded domain /11/: let $D\subset E_3$ be a bounded domain whose boundary ∂D is a finite union of Lyapunov surfaces /9/; then any SLP with density in $C(\partial D)$ is a function in $C^{0,\alpha}(E_3)$, $0<\alpha<1$; a DLP with density in $C(\partial D)$ is a function in $C^{0,\alpha}(\partial D)$, $0<\alpha<1$.

On this basis we can prove the following

Lemma 1. The ranges of the operators $\sum\limits_n R_n'$, $\sum\limits_n R_n$ are dense in $L_2(S_\Delta)$.

Proof. For sufficiently smooth BE $\Delta s_n\subset S_\Delta$, linear combinations of potentials

$$\sum_n\gamma_n(x),\quad \sum_n\beta_n(x),\quad x\in S_\Delta$$

with densities (1.8), (1.7), respectively, in $C(S_\Delta)$ generate operators which, because of the properties of the potentials indicated in the previous paragraph, map $C(S_\Delta)$ into $C^{0,\alpha}(S_\Delta)$, $0<\alpha<1$; the functions thus obtained are uniformly continuous in the finite domain S_Δ and therefore form a dense subset of $L_2(S_\Delta)$ (see, e.g., /12/).

The basis property of the sequence of potentials (1.9) may be derived from the following propositions.

Lemma 2. Let the densities of the potentials (1.1) be sequences of linearly independent functions. Then (1.9) are also sequences of linearly independent functions.

Proof. Suppose the contrary: there exist constants $\Phi_{1k},\Phi_{2k},\ldots,\Phi_{Nk}$, not all zero, such that (see (1.5))

$$R\Big(\sum_k\Phi_{1k}\psi_k\Big)+R\Big(\sum_k\Phi_{2k}\psi_k\Big)+\ldots+R\Big(\sum_k\Phi_{Nk}\psi_k\Big)=R\Big(\sum_n\sum_k\Phi_{nk}\psi_k\Big)=0$$

Since $R(0)=0$ (see (1.11)), this implies that the double sum in parentheses in the last equality vanishes, contrary to the assumption that the densities of the potentials $\{\beta_n\}$ are linearly independent; the linear dependence of the sequence $\{\gamma_n\}$ is proved in analogous fashion.

For the sequel, we first present a more precise definition of the completeness in $L_2(S)$ of a sequence $\{\varphi_n\}$ of interpolating functions of type (1.5) (or (1.6)). The definition of completeness for a system of coordinate functions in a Hilbert space H that is usually used in connection with the Ritz procedure of approximation /13/ runs as follows: for any function $\varphi \in H$ and any number $\varepsilon > 0$, there exist a natural number M and constants $\alpha_1, \alpha_2, \ldots, \alpha_M$ such that

$$\|\varphi - \sum_n \alpha_n \varphi_n\|_H < \varepsilon. \tag{1.12}$$

We now need a more discriminating definition, since we are now dealing with approximation by piecewise-constant interpolation functions (similar to splines), in which case we have a more precise estimate, commonly used in the FEM /14-16/:

$$\|\varphi - \sum_n \sum_k \Phi_{nk} \psi_k\|_{L_2(S_\Delta)} = O(\bar{d}^\delta) \geqslant c_0 \underline{d}^{p+1} \tag{1.13}$$

$$\bar{d} = \max_n \operatorname{diam} \Delta s_n, \quad \underline{d} = \min_n \operatorname{diam} \Delta s_n;\ \delta, c_0 > 0$$

provided that the finite elements Δs_n and function φ are sufficiently smooth.

Thus, for a given boundary (1.2) the number M in (1.12) depends on the number N of BEs Δs_n, and the number ε cannot be less than $c_0 d^{p+1}$, where p is the order of the polynomial.

Lemma 3. The sequences of linearly independent potentials (1.9) are complete in $L_2(S_\Delta)$.

Proof. As far as the sequence $\{\beta_n\}$ is concerned, the assertion is trivial, since $\beta_n|_{S_\Delta} = \varphi_n, \forall n$ (see (1.7)) and the interpolating functions (1.5) are complete in the sense of (1.13). In view of (1.8) and the completeness of the interpolating functions (1.6) in that sense, the same conclusion is true for $\{\partial_{v_n} \gamma_n\}$. It remains, therefore, to show that the sequence $\{\gamma_n\}$ is complete in $L_2(S_\Delta)$.

For a set of uniformly continuous functions which is dense in $L_2(S_\Delta)$ (see Lemma 1) this completeness is equivalent to R'-completeness of the sequence $\{t_n\}$ in the sense of (1.13) (for the definition of A-completeness, see /13, p.453/), since we can choose a number N' and constants $T_{1k}', T_{2k}', \ldots, T_{N'k}'$ so that

$$\left\| t - \sum_{n=1}^{N'} \sum_{k=1}^{K} T'_{nk} \psi_k' \right\|_{L_2(S_\Delta)} < \frac{\varepsilon'}{2c'}, \quad \varepsilon' > 0$$

where c' is the constant in (1.10); then for any SLP $\gamma' = R't$ from the above dense subset of $L_2(S_\Delta)$ and the SLP

$$\gamma_{N'} = R'\Big(\sum_{n=1}^{N'} t_n\Big) = R'\Big(\sum_{n=1}^{N'} \sum_k T'_{nk} \psi_k'\Big)$$

it follows from (1.10) that

$$\|\gamma' - \gamma_{N'}\|_{L_2(S_\Delta)} \leqslant c' \left\| t - \sum_{n=1}^{N'} \sum_k T'_{nk} \psi_k' \right\|_{L_2(S_\Delta)} < c' \frac{\varepsilon'}{2c'} = \frac{\varepsilon'}{2} \tag{1.14}$$

Now let γ be an arbitrary SLP in $L_2(S_\Delta)$; there exists a SLP γ' in a dense subset of $L_2(S_\Delta)$ such that

$$\|\gamma - \gamma'\|_{L_2(S_\Delta)} < \frac{\varepsilon'}{2}$$

Choose the number N' and constants T'_{nk} to satisfy (1.14). Then by the triangle inequality

$$\|\gamma - \gamma_{N'}\|_{L_2(S_\Delta)} \leqslant \|\gamma - \gamma'\|_{L_2(S_\Delta)} + \|\gamma' - \gamma_{N'}\|_{L_2(S_\Delta)} < \varepsilon' \tag{1.15}$$

thus proving the completeness to the sequence $\{\gamma_n\}, n = 1, \ldots, N'$, in $L_2(S_\Delta)$.

2. In order to use the Ritz process to construct solutions of variational problems for BFs which are equivalent to second-order elliptic boundary-value problems, we have to construct basis systems of functions in the appropriate boundary "energy" spaces /17/.

The minimization of BFs of the form

$$f(u) = \int_S u\partial_\nu u\, ds - 2\int_S ug\, ds \tag{2.1}$$

with respect to all solutions of a differential equation $Au = 0$ in G (on the assumption that $\int udG = 0$ if the quadratic form $B_A(u, u)$ is only non-negative) is equivalent to a non-homogeneous Neumann problem (g is a given function), and the energy space of admissible functions is the trace subspace on $S, W_2^{*1/2}(S) \subset W_2^{1/2}(S)$, with scalar product

$$[u, v]_{1/2, S} = \int_S u\partial_\nu v\, ds, \quad |u|_{1/2, S} = \{[u, u]_{1/2, S}\}^{1/2} \tag{2.2}$$

(see /17/). We then have continuous and dense embeddings /17/

$$W_2^{*1/2}(S) \subset L_2(S) \subset W_2^{-1/2}(S) \tag{2.3}$$

and in this situation Riesz's theorem on the general form of a continuous linear functional in a Hilbert space /17/ holds:

$$[u, v]_{1/2, S} = (u, Nv)_{0, S} = (Nu, Nv)_{-1/2, S}, \ \forall u, v \in W_2^{*1/2}(S) \tag{2.4}$$

where N is the isometry of $W_2^{*1/2}(S)$ onto $W_2^{-1/2}(S)$ generated by the normal derivative operator on S,

$$\langle u, \partial_\nu v\rangle = (u, Nv)_{0, S} \tag{2.5}$$

Here $\langle\, ,\rangle$ is the duality relation on $W_2^{1/2}(S) \times W_2^{-1/2}(S)$, $W_2^{1/2}(S)$ is the Sobolev-Slobodetskii space and $W_2^{-1/2}(S)$ is its dual; $(\, ,)_{0, S}$, $(\, ,)_{-1/2, S}$ are the scalar products in $L_2(S)$, $W_2^{-1/2}(S)$, respectively.

Lemma 4. The sequences of potentials (1.9) are complete in $W_2^{*1/2}(S_\Delta)$.

Proof. By construction, $W_2^{*1/2}(S_\Delta)$ is the subspace of traces of functions which satisfy the equation $Au = 0$ in G_Δ (and perhaps also the condition $\int udG_\Delta = 0$). Consequently, the traces $\beta_n|_{S_\Delta}, \gamma_n|_{S_\Delta}$ of the potentials $\beta_n(x), \gamma_n(x), x \in G_\Delta$, which satisfy the above conditions, belong to $W_2^{*1/2}(S_\Delta)$. By (2.5), the elements $\partial_{\nu_n}\beta_n$ and $\partial_{\nu_n}\gamma_n$ may be identified with $N\beta_n$ and $N\gamma_n$, respectively /17/. For any elements $N\beta^\varepsilon$, $N\gamma^\varepsilon$ in $L_2(S_\Delta)$, which is dense in $W_2^{-1/2}(S_\Delta)$ (see (2.3)), we have the following inequality (similar to (1.14)):

$$\|N\gamma^\varepsilon - N\gamma_{N^\varepsilon}\|_{-1/2, S_\Delta} \leqslant \|\gamma^\varepsilon - \gamma_{N^\varepsilon}\|_{1/2, (S_\Delta)} < \varepsilon''/2 \tag{2.6}$$

Hence, for any elements $N\beta, N\gamma \in W_2^{-1/2}(S_\Delta)$ we obtain, as in (1.15),

$$\|N\gamma - N\gamma_{N^\varepsilon}\|_{-1/2, S_\Delta} < \varepsilon''$$

Consequently, the sequence $\{N\gamma_n\}$, $n = 1, \ldots, N''$, is complete in $W_2^{-1/2}(S_\Delta)$; a similar assertion is true for $\{N\beta_n\}, n = 1, \ldots, N''$. Finally, the equality

$$\|\partial_\nu u\|_{-1/2, S} = \|Nu\|_{-1/2, S} = |u|_{1/2, S}$$

which follows from (2.4), implies that

$$|\gamma - \gamma_{N^\varepsilon}|_{1/2, S_\Delta} = \|N\gamma - N\gamma_{N^\varepsilon}\|_{-1/2, S_\Delta} < \varepsilon'' \tag{2.7}$$

proving the lemma.

If $u, v \in W_2^{*1/2}(S)$, Green's formula and (2.2) imply the equality

$$[u, v]_{1/2, S} = B_A(u, v) = [u, v]_{H_A} \tag{2.8}$$

and the energy scalar product is equivalent to the scalar product in $W_2^1(G)$. It then follows from (2.7) and (2.8) that the sequences of potentials (1.1) are complete with respect to the energy of the Neumann problem (and also in the norm of $W_2^1(G_\Delta)$).

From the sum total of what we have proved it follows that BE-approximations to the solution of the Neumann problem can be constructed in terms of the potentials $\{\beta_n\}$, $\{\gamma_n\}$, treating the problem as that of minimizing the energy integral

$$F(u) = [u, u]_{H_A} - 2\int_S ug\, ds, \quad u \in H_A \tag{2.9}$$

Based on Lemma 4 one can prove that the "Ritz" BE-approximations to a function minimizing the BF (2.1) are convergent; the same applies to the minimization problem for (2.9). By construction (see /1, 2/), as already pointed out, the Ritz BE-approximations to the solution of a variational problem for BFs (2.1) are identically equal to the sequence of DLPs defined by the first formula of (1.1) with an unknown density of type (1.5) (direct formulation):

$$u_N \equiv \sum_n \beta_n(x), \quad x \in G_\Delta \tag{2.10}$$

or to the sequence of SLPs defined by the second formula of (1.1) with an unkown density of type (1.6) (dual formulation):

$$u_N \equiv \sum_n \gamma_n(x), \quad x \in G_\Delta \tag{2.11}$$

In the first case the unknowns are the values of u at the nodes of the discrete boundary; in the second, they are the values of the normal derivative $\partial_\nu u$ at the nodes of S_Δ. The nodal values are determined from the appropriate systems of Ritz BE-equations for BFs of type (2.1) /1, 2/.

Thus, the potential sequences (1.9) (for $x \in G_\Delta$) are minimizing sequences for BFs (2.1), i.e.,

$$\lim_{N\to\infty} f_\Delta(u_N) = f(u_0) = \min_{u \in W_2^{*1/2}(S)} f(u) \tag{2.12}$$

Here f_Δ is a functional approximating f over the approximations (2.10) (or (2.11)) and

$$f(u_0) = -\int_S u_0 \partial_\nu u_0 ds = -|u_0|^2_{1/2, S}$$

where u_0 is the exact solution of the problem $\min_u f(u)$ (see /17/).

We know (see /13, p.89/ and also /17/) that

$$f_\Delta(u_N) = |u_0 - u_N|^2_{1/2, S_\Delta} - |u_0|^2_{1/2, S_\Delta}$$

Hence, by (2.12), we obtain convergence (as $S_\Delta \to S$)

$$\lim_{N\to\infty} |u_0 - u_N|_{1/2, S} = 0 \tag{2.13}$$

We can now prove the following theorem.

Theorem. BE-solution of the problem

$$\min_{u_N} f_\Delta(u_N), \quad u_N \in W_2^{*1/2}(S_\Delta)$$

is equivalent to the solution of the Neumann problem

$$Au_0(x) = 0, \ x \in G, \ \partial_\nu u_0 |_S = g$$

in the sense of convergence as $\operatorname{diam} \Delta s_n \to 0 \Rightarrow S_\Delta \to S$ (or, if $S_\Delta \equiv S$)

$$\lim_{N\to\infty} \| u_0 - u_N \|_{W_2^1(G)} = 0$$

$$\lim_{N\to\infty} \| g - \partial_{\nu_\Delta} u_N \|_{-1/2, S} = 0$$

The first convergence relation follows from (2.8), the equivalence of the energy norm and the norm in $W_2^1(G)$, and the convergence (2.13). The second follows from (2.4a), (2.13) and the boundary condition of the Neumann problem.

Using scalar potentials, one can extend the above results on the convergence of the BE-approximations to the case of vector potentials in the appropriate vector-function spaces.

Thus, we have also established the validity of the BE-approximations to the variational solutions obtained in /1, 2/ for problems in linear elasticity theory.

The convergence of BE-approximations to solutions of minimization problems for GTFs /1, 2/ can also be proved by using the basis property of a sequence of boundary potentials, as established above, combined with known convergence properties of the Ritz process as applied to the minimization of GTFs /13, 18, 19/.

In conclusion, we note that the variational formulations of the BEM presented in /1, 2/ recall the approach used in /20/ in connection with various applied mechanical problems, which places the BEM in a complex function-theoretic setting and employs Cauchy integrals to link the boundary values of the approximating function with its values in the interior of the domain.

REFERENCES

1. TERESHCHENKO V.YA., On some formulations of the method of boundary elements. Prikl. Mat. Mekh., 51, 4, 1987.
2. TERESHCHENKO V.YA., Dual formulations of the boundary element method. Application to problems of elasticity theory for inhomogeneous bodies. Prikl. Mat. Mekh., 55, 1, 1991.
3. BANNERJEE P.K. and BUTTERFIELD R., Boundary Element Methods in the Applied Sciences, Mir, Moscow, 1984.
4. NORRIE D.H. and DE FRIES G., Introduction to the Finite Element Method, Mir, Moscow, 1981.
5. GALLAGHER R.H., Finite Element Analysis Fundamentals. Prentice-Hall, Englewood Cliffs, N.J., 1975.
6. ZIENKIEWICZ O.C. and MORGAN K., Finite Elements and Approximation. Wiley, New York, 1983.
7. COURANT R., Partial Differential Equations, Mir, Moscow, 1964.
8. SOBOLEV S.L., Some Applications of Functional Analysis in Mathematical Physics, Nauka, Moscow, 1988.
9. GYUNTER N.M., Potential Theory and its Application to Basic Problems of Mathematical Physics, Gostekhizdat, Moscow, 1953.
10. MIKHLIN S.G., Multidimensional Singular Integrals and Integral Equations, Fizmatgiz, Moscow, 1962.
11. MIKHLIN S.G., A Course of Mathematical Physics, Nauka, Moscow, 1968.
12. NATANSON I.P., Theory of Functions of a Real Variable, Gostekhizdat, Moscow-Leningrad, 1950.
13. MIKHLIN S.G., Variational Methods in Mathematical Physics, Nauka, Moscow, 1970.
14. VARGA R., Functional Analysis and Approximation Theory in Numerical Analysis, SIAM, Philadelphia, 1971.
15. CIARLET P.,The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1977.
16. KORNEYEV V.G., High-Accuracy Schemes of the Finite Element Method, Izd. Leningrad. Gos. Univ., Leningrad, 1977.
17. TERESHCHENKO V.YA., Orthogonal expansions on the boundary of the domain in elliptic boundary-value problems. Differents. Uravn., 18, 3, 1982.
18. TERESHCHENKO V.YA., Generalization of Trefftz's method for three-dimensional problems of elasticity theory. Zh. Vychisl. Mat. mat. Fiz., 16, 4, 1976.
19. TERESHCHENKO V.YA., On the Ritz process in constructing approximate solutions of problems of elasticity theory by the generalized Trefftz method. Zh. Vychisl. Mat. mat. Fiz., 20, 4, 1980.
20. GROMADKA T. and LEY CH., The Complex Boundary Element Method in Engineering Problems, Mir, Moscow, 1990.

Translated by D.L.